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Completions of generalized inverse $*$-semigroups[1]

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Abstract

It is well known that every inverse semigroup can be embedded both in a (join) complete inverse semigroup and a meet complete inverse semigroup (see [8]). The purpose of this paper is to obtain its generarization for generalized inverse $*$-semigroups. We succeed the former, that is, each generalized inverse $*$-semigroup S is embedded in a $*$-complete, infinetely distributive generalized inverse $*$-semigroup. Unfortunately, we can not answer for the later. However, we have that S is embedded in $K(S)$ consisting of all cosets of S.

1 Preliminaries

A semigroup S with a unary operation $* : S \to S$ is called a *regular $*$-semigroup* if it satisfies

$$\text{(i) } (x^*)^* = x; \qquad \text{(ii) } (xy)^* = y^*x^*; \qquad \text{(iii) } xx^*x = x.$$

Let S be a regular $*$-semigroup. An idempotent e in S is called a *projection* if $e^* = e$. For a subset A of S, denote the sets of idempotents and projections of A by $E(A)$ and $P(A)$, respectively.

Let S be a regular $*$-semigroup. If $E(S) = P(S)$, S is called an *inverse semigroup.* If eSe is an inverse subsemigroup of S for any $e \in E(S)$, it is called a *locally inverse $*$-semigroup.* If $E(S)$ forms a subsemigroup of S, it is called an *orthodox $*$-semigroup.* If S is orthodox and locally inverse, it is called a *generalized inverse $*$-semigroup.* It is well known that S is a generalized inverse $*$-semigroup if and only if $E(S)$ satisfies the identity $xyzw = xzyw$.

Result 1.1. [4] *Let S be a regular $*$-semigroup. Then we have*

(1) $E(S) = P(S)^2$.

(2) *For any $a \in S$ and $e \in P(S)$, $a^*ea \in P(S)$.*

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(3) *Each $\mathcal{L}$-class and $\mathcal{R}$-class contains one and only one projection.*

Let S be a regular $*$-semigroup. Define a relation $\leqq$ on S as follows:

$$a \leqq b \iff a = eb = bf \quad \text{for some} \quad e, f \in P(S).$$

Result 1.2. [5] *Let a and b be elements of a regular $*$-semigroup S. Then the following statements are equivalent:*

(1) $a \leqq b$,

(2) $aa^* = ba^*$ *and* $a^*a = b^*a$,

(3) $aa^* = ab^*$ *and* $a^*a = a^*b$,

(4) $a = aa^*b = ba^*a$.

The relation $\leqq$, defined above, is a partial order on S which preserves the $*$-operation. We call $\leqq$ *the natural order* on S. It is well known that S is a locally inverse $*$-semigroup if and only if $\leqq$ is compatible.

Proposition 1.3. *Let S be a regular $*$-semigroup. Then S is a generalized inverse $*$-semigroup if and only if $xey \leqq xy$ for any $x, y \in S$ and $e \in P(S)$*

Let $(P, \leqq)$ be a partial order set. A subset Q of P is said to be an *order ideal* if $x \leqq y \in Q$ implies $x \in Q$. For $x \in P$, $[x] = \{y \in P : y \leqq x\}$ is the smallest order ideal of P containing x, which is called *the principal order ideal* of P containing x.

Proposition 1.4. *Let S be a regular $*$-semigroup. Then $P(S)$ is an order ideal of S. Moreover, if S is orthodox, then $E(S)$ is an order ideal.*

Let S and T be regular $*$-semigroups. A mapping $\theta : S \to T$ is called a $*$-*homomorphism* if, for any $a, b \in S$,

$$\theta(ab) = \theta(a)\theta(b) \quad \text{and} \quad \theta(a^*) = \theta(a)^*.$$

The following properties are well known.

Result 1.5. *Let $\theta : S \to T$ be a $*$-homomorphism between regular $*$-semigroups.*

(1) *If $e \in E(S)$, then $\theta(e) \in E(T)$.*

(2) *If $e \in P(S)$, then $\theta(e) \in P(T)$.*

(3) *If U is a regular $*$-subsemigroup of S, then $\theta(U)$ is a regular $*$-subsemigroup of T.*

(4) *If V is a regular $*$-subsemigroup of T, then $\theta^{-1}(V)$ is a regular $*$-subsemigroup of S.*

(5) *The mapping θ is order-preserving.*

The notation and terminology are those of [7] and [8], unless otherwise stated.

2 $*$-Compatibility relations and infinitely distributive semigroups

Let S be a regular $*$-semigroup. For any $s, t \in S$, *the left $*$-compatibility relation* is defined by

$$s \sim_l^* t \Leftrightarrow st^* \in P(S),$$

the right $*$-compatibility relation is defined by

$$s \sim_r^* t \Leftrightarrow s^*t \in P(S),$$

and *the compatibility relation* is defined by

$$s \sim^* t \Leftrightarrow s^*t, st^* \in P(S).$$

A subset A of S is said to be *$*$-compatible* if $a \sim^* b$ for all $a, b \in A$.

Lemma 2.1. *Let S be a regular $*$-semigroup and let $s, t \in S$. Then $s \sim^* t$ if and only if the greatest lower bound $s \wedge t$ of s and t exists and*

$$s \wedge t = st^*t = ts^*t = ts^*s = st^*s = ss^*t = tt^*s.$$

Lemma 2.2. *Let S be a locally inverse $*$-semigroup, and let $s, t, u, v \in S$. Then*

(1) *$s \leqq t, u \leqq v$ and $t \sim^* v$ implies that $s \sim^* u$.*

(2) *$[s]$ is a $*$-compatible order ideal of S.*

If S is a generalized inverse $*$-semigroup, then

(3) *$s \sim^* t$ and $u \sim^* v$ implies that $su \sim^* tv$.*

Lemma 2.3. *Let S be a locally inverse $*$-semigroup and let A and B be non-empty subsets of projections and idempotents, respectively. Then we have the following:*

(1) *If $\bigwedge A$ exists, then it is a projection.*

(2) *If $\bigvee A$ exists, it is a projection.*

Moreover, let S be a generalized inverse $*$-semigroup. Then

(3) *If $\bigwedge B$ exists, it is an idempotent.*

(4) *If $\bigvee B$ exists, it is an idempotent.*

Lemma 2.4. *Let S be a locally inverse $*$-semigroup and let A be a non-empty subset of S such that $\bigvee A$ exists. Then any two elements of A are $*$-compatible.*

A regular $*$-semigroup is said to be *left infinitely distributive* if, whenever A is a non-empty subset of S for which $\bigvee A$ exists, then $\bigvee sA$ exists for any element $s \in S$ and $s(\bigvee A) = \bigvee sA$. *Right infinitely distributive* is defined analogously. Also a semigroup which is both left and right infinitely distributive is called *infinitely distributive*. We say that a regular $*$-semigroup is *$*$-complete* if every its non-empty $*$-compatible subset has a join.

Proposition 2.5. *Let S be a locally inverse $*$-semigroup and $A = \{a_i : i \in I\}$ a non-empty subset of S.*

(1) *If $\bigvee a_i$ exists then $\bigvee a_i^* a_i$ exists and $(\bigvee a_i)^*(\bigvee a_i) = \bigvee a_i^* a_i$.*

(2) *If $\bigvee a_i$ exists then $\bigvee a_i a_i^*$ exists and $(\bigvee a_i)(\bigvee a_i)^* = \bigvee a_i a_i^*$.*

Theorem 2.6. *Let S be a infinitely distributive locally inverse $*$-semigroup. If A and B are non-empty subsets of S such that $\bigvee A$, $\bigvee B$ and $\bigvee AB$ exist, then $\bigvee AB = (\bigvee A)(\bigvee B)$.*

3 Join completions

Let A be a subset of a regular $*$-semigroup S. It is said to be $*$-*permissible* if it is a $*$-compatible order ideal of S. The set of all $*$-permissible subsets of S is denoted by $C^*(S)$.

Lemma 3.1. *Let S be a regular $*$-semigroup and A its $*$-permissible subset. Then*

$$A^*A = \{a^*a : a \in A\} \quad \textit{and} \quad AA^* = \{aa^* : a \in A\}$$

are both order ideals.

Lemma 3.2. *Let S be a regular $*$-semigroup. If A is a $*$-permissible subset of S which satisfies $AA = A$, then it is a subset of $E(S)$. Moreover, A satisfies $A^* = A$, it is a subset of $P(S)$.*

Now, we have the main theorem.

Theorem 3.3. *Let S be a generalized inverse $*$-semigroup. Then $C^*(S)$ is a $*$-complete, infinitely distributive generalized inverse $*$-semigroup. And the mapping $\iota : S \to C^*(S)$ $(s \mapsto [s])$ is an injective $*$-homomorphism. Moreover, every element of $C^*(S)$ is a join of non-empty subset of $\iota(S)$.*

Theorem 3.4. *If $\theta : S \to T$ be a $*$-homomorphism to a $*$-complete, infinitely distributive generalized inverse $*$-semigroup, then there exists a unique join-preserving $*$-homomorphism $\phi : C^*(S) \to T$ such that $\phi\iota = \theta$.*

Now we can obtain that the category of $*$-complete, infinitely distributive generalized inverse $*$-semigroups together with join-preserving $*$-homomorphisms is a reflective subcategory of the category of generalized inverse $*$-semigroups and $*$-homomorphism.

Theorem 3.5. *The function $S \mapsto C^*(S)$ is the object part of a functor from the category of generalized inverse $*$-semigroups and $*$-homomorphisms to the category of $*$-complete, infinitely distributive generalized inverse $*$-semigroups and join-preserving $*$-homomorphisms.*

4 Cosets of generalized inverse $*$-semigroups

Let S be a regular $*$-semigroup and X its subset. We call

$$[X]^{\uparrow} = \{s \in S : x \leqq s \ \text{ for some } \ x \in X\}$$

the closure of X in S. If $X = \{x\}$ consists a single element, we denote it by $[x]^{\uparrow}$, which is called *the principal closure* containing x. A subset is said to be *closed* if it is equal to its closure.

Let S be a generalized inverse $*$-semigroup. A non-empty subset H of S is called a *coset* if HH^*H, and the set of all cosets of S is denoted by $K(S)$. We first remark to justify the use of the term coset.

Proposition 4.1. *Let A be a non-empty subset of a group G. Then $A = AA^*A$ $(= aa^{-1}A)$ if and only if A is a coset of a subgroup of G.*

A further justification for the term coset comes from the theory of representation of generalized inverse $*$-semigroups.

Proposition 4.2. *Let $\theta : S \to \mathcal{GI}_{(X;\Omega)}$ be a representation of a generalized inverse $*$-semigroup S. Let $x, y \in X$ and put $H_{x,y} = \{s \in S : \theta(s)(x) = y\}$. Then if $H_{x,y}$ is non-empty, it is a coset.*

We give another characterization of cosets in the sense of Dubreil [1]. For non-empty subsets A and B of a semigroup S, define

$$A.\cdot B = \{s \in S : Bs \subseteqq A\} \ \text{ and } \ A\cdot .B = \{s \in S : sB \subseteqq A\}.$$

If $B = \{b\}$, we denote each by $A.\cdot B$ and $A \cdot .B$.

Lemma 4.3. *Let S be a generalized inverse $*$-semigroup. If A is a coset and $A.\cdot B$ $[A\cdot .B]$ is a non-empty subset of S, then $A.\cdot B$ $[A\cdot .B]$ is a coset.*

Theorem 4.4. *Let H be a non-empty subset of a generalized inverse $*$-semigroup S. Then the following statements are equivalent:*

(1) *H is a coset,*

(2) $H.\cdot s \cap H.\cdot t \neq \emptyset \ \Rightarrow \ H.\cdot s = H.\cdot t$ for any $s, t \in S^1$,

(3) $H\cdot .s \cap H\cdot .t \neq \emptyset \ \Rightarrow \ H\cdot .s = H\cdot .t$ for any $s, t \in S^1$,

(4) $xu, vu, vy \in H \ \Rightarrow \ xy \in H$ for any $x, y \in S$ and $u, v \in S^1$.

Let S be a generalized inverse $*$-semigroup. We now introduce a new binary operation on $K(S)$ and it becomes a generalized inverse $*$-semigroup with respect to the operation. It is clear that the intersection of any non-empty set of cosets is either empty or a coset. For a non-empty subset X of S, we define $j(X)$ to be the intersection of all cosets containing X, that is, the smallest coset containing X. Define a binary operation $\otimes$ and a unary operation $*$ on S as follows:

$$A \otimes B = j(AB) \ \text{ and } \ (A)^* = A^*.$$

Theorem 4.5. *Let S be a generalized inverse $*$-semigroup. Then $K(S)(\otimes, *)$ is a generalized inverse $*$-semigroup.*

Proposition 4.6. *Let S be a generalized inverse $*$-semigroup and $s \in S$. Then $[s]^\dagger$ is a coset.*

Proposition 4.7. *Let S be a generalized inverse $*$-semigroup. Then, for any $A, B \in K(S)$,*

$$A \leqq B \quad \Rightarrow \quad A \supseteqq B.$$

Now, we can immediately obtain the following theorem.

Theorem 4.8. *Let S be a generalized inverse $*$-semigroup. Then the mapping $\iota : S \to K(S)$ $(s \mapsto [s]^\dagger)$ is an injective $*$-homomorphism, and each element of $K(S)$ is the union of a non-empty subset of $\iota(S)$.*

Remark. We showed that, for $A, B \in K(S)$, $A \leqq B$ implies $A \supseteqq B$ in Proposition 4.7. However, we do not know where the converse is true or not. If it is true, we can change "the union " to "the meet" in Theorem 4.8.

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